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Blow-up rate for a nonlinear diffusion equation[☆]

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Abstract

In this work we study the blow-up rate for a nonlinear diffusion equation with an inner source and a nonlinear boundary flux, which is equivalent to a porous medium equation with convection. Depending upon the sign of a parameter included, the source can be positive or negative (absorption). By the scaling method, we obtain that the blow-up rate is independent of a negative source, while for the situation with a positive source, the blow-up rate is determined by the interaction between the inner source and the boundary flux. Comparing with the previous results for the porous medium model without convection, we observe that the gradient term included here does not affect the blow-up rates of solutions.

Keywords: Blow-up rate; Nonlinear diffusion; Porous medium equation; Convection; Nonlinear boundary flux; Absorption

1. Introduction

In this work, we consider the following nonlinear diffusion equation:

$$\begin{cases} w_t = \left(e^{(m-1)w}\right)_{xx} - \lambda e^{(p-1)w}, & (x,t) \in (0,1) \times (0,T), \\ w_x(0,t) = 0, \quad w_x(1,t) = e^{(q-m)w(1,t)}, & t \in (0,T), \\ w(x,0) = w_0(x), & x \in [0,1], \end{cases} \tag{1.1}$$

where we have parameters $q > m > 1$, $p > 1$, $\lambda \neq 0$; w_0 is nonnegative and satisfies the compatibility condition. On setting $u = e^w$, (1.1) becomes a porous medium equation of the form

$$\begin{cases} u_t = \dfrac{m-1}{m}(u^m)_{xx} - (m-1)u^{m-2}|u_x|^2 - \lambda u^p, & (x,t) \in (0,1) \times (0,T), \\ (u^m)_x(0,t) = 0, \quad (u^m)_x(1,t) = mu^q(1,t), & t \in (0,T), \\ u(x,0) = e^{w_0}, & x \in [0,1]. \end{cases} \tag{1.2}$$

Eq. (1.2) can be interpreted as a slow diffusion equation that describes, e.g., heat propagation in nonlinear media with nonlinear diffusion, inner nonlinear absorption (sources), nonlinear convection, and nonlinear boundary flux. For

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$\lambda > 0$, the global existence and nonexistence of solutions to (1.1) have been studied in [1]. It was shown that the blow-up phenomenon occurs if $p < 2q - m$, or $p = 2q - m$ with $\lambda < (m-1)(q-1)$ for large initial data. For $\lambda < 0$, a simple computation shows that $\underline{w} = \log[(1 + (p-1)\lambda t)]^{-1/(p-1)}$ is a subsolution of (1.1), which implies that every solution of (1.1) blows up in finite time. In summary, we have the following proposition:

Proposition 1. (i) *Let $\lambda > 0$. The solution of* (1.1) *is global if $p > 2q - m$, and blows up in a finite time if either $p < 2q - m$, or $p = 2q - m$ with $\lambda < (m-1)(q-1)$ for large initial data.* (ii) *Let $\lambda < 0$. The solution of* (1.1) *blows up in a finite time for any nonnegative initial data.* □

We are interested in the blow-up rate for the problem (1.1). In [2], Rossi obtained the blow-up rate for a semilinear parabolic equation of the form

$$\begin{cases} u_t = u_{xx} - \lambda u^p & \text{in } (0,1) \times (0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = u^q(1,t) & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{on } [0,1], \end{cases} \tag{1.3}$$

where $p, q > 1$. For $\lambda > 0$ it was shown that if $p < 2q - 1$ or $p = 2q - 1$ with $\lambda < q$, then $\max_{[0,1]} u(\cdot,t) = O((T-t)^{-1/2(q-1)})$, while for $\lambda < 0$, the blow-up rate is $O((T-t)^{-1/2(q-1)})$ if $p \le 2q - 1$, or $O((T-t)^{-1/(p-1)})$ if $p > 2q - 1$.

Recently, Jiang et al. [3,4] studied the corresponding nonlinear diffusion case

$$\begin{cases} u_t = (u^m)_{xx} - \lambda u^p & \text{in } (0,1) \times (0,T), \\ (u^m)_x(0,t) = 0, \quad (u^m)_x(1,t) = u^q(1,t) & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{on } [0,1] \end{cases} \tag{1.4}$$

and obtained for $\lambda > 0$, $q > m$ that, if $p < 2q - m$, or $p = 2q - m$ with $\lambda < q/m$, then $\max_{[0,1]} u(\cdot,t) = O((T-t)^{-1/(2q-m-1)})$. When $\lambda < 0$, the blow-up rate is $O((T-t)^{-1/(2q-m-1)})$ if $p \le 2q - m$ with $q > m$, or $O((T-t)^{-1/(p-1)})$ if $p > 2q - m$ with $p > m$. One can refer to, e.g., [5–9], for similar work concerning blow-up rates.

The main results of this work are the following blow-up rate estimate theorems.

Theorem 1. *Let w be the solution of* (1.1) *with $\lambda > 0$, $\left(e^{(m-1)w_0}\right)_{xx} - \lambda e^{(p-1)w_0} \ge 0$ on $[0,1]$. If $p < 2q - m$, or $p = 2q - m$ with $\lambda < (m-1)(q-1)$, then*

$$\log c(T-t)^{-\frac{1}{2q-m-1}} \le \max_{[0,1]} w(\cdot,t) \le \log C(T-t)^{-\frac{1}{2q-m-1}}$$

for $t \in (0,T)$ and positive constants c, C.

Theorem 2. *Let w be the solution of* (1.1) *with $\lambda < 0$, $w_0' \ge 0$, $\left(e^{(m-1)w_0}\right)_{xx} - \lambda e^{(p-1)w_0} \ge 0$ on $[0,1]$. Then*

$$\log c(T-t)^{-\frac{1}{2q-m-1}} \le \max_{[0,1]} w(\cdot,t) \le \log C(T-t)^{-\frac{1}{2q-m-1}} \quad \text{if } p \le 2q - m,$$

$$\log c(T-t)^{-\frac{1}{p-1}} \le \max_{[0,1]} w(\cdot,t) \le \log C(T-t)^{-\frac{1}{p-1}} \quad \text{if } p > 2q - m$$

for $t \in (0,T)$ and positive constants c, C.

Remark 1. The blow-up rate for (1.1) with $\lambda > 0$ obtained in Theorem 1 is equivalent to $O((T-t)^{-\frac{1}{2q-m-1}})$ for u in (1.2) since $u = e^w$. Similarly, it follows from Theorem 2 with $\lambda < 0$ that the blow-up rate for u in (1.2) is $O((T-t)^{-\frac{1}{2q-m-1}})$ when $p \le 2q - m$, or $O((T-t)^{-\frac{1}{p-1}})$ if $p > 2q - m$. The above blow-up rate with $\lambda > 0$ depends only on the nonlinear boundary flux if $p \le 2q - m$. When $\lambda < 0$, the blow-up rate is dominated by either the boundary flux if $p \le 2q - m$, or the inner source if $p > 2q - m$. In particular, these results for the solution u of (1.2) agree with those for the similar model (1.4) without convection obtained in [3,4], namely, the gradient term in (1.2) makes no contribution to the blow-up rate. In other words, the convection here is insufficient to affect the result of the interaction between the boundary flux and the inner source in (1.2).

2. Blow-up rates

In this section, we will establish the blow-up rates for (1.1) to prove Theorems 1 and 2. The main tool used here is a scaling argument [8,10–12].

Proof of Theorems 1 and 2. Let w be a solution of (1.1) with blow-up time T. For $\lambda > 0$, since $\left(e^{(m-1)w_0}\right)_{xx} - \lambda e^{(p-1)w_0} \geq 0$ on [0,1], the maximum principle yields $w_t \geq 0$, and hence $\left(e^{(m-1)w}\right)_{xx} > 0$, namely, $\left(e^{(m-1)w}\right)_x$ is nondecreasing with respect to x. Taking this together with the boundary condition $w_x(0,t) = 0$, we know that $w_x \geq 0$ for $(x,t) \in [0,1] \times [0,T)$. When $\lambda < 0$, it is easy to see that $w_x, w_t \geq 0$ since $w_0' \geq 0$, $\left(e^{(m-1)w_0}\right)_{xx} - \lambda e^{(p-1)w_0} \geq 0$ by the maximum principle. Define

$$M(t^*) = w(1,t^*) = \max_{[0,1]} w(\cdot, t^*), \quad t^* \in (0,T). \tag{2.1}$$

At first consider the case $p < 2q - m$ in Theorems 1 and 2. Define

$$\varphi_a(y,s) = e^{(m-1)[w(ay+1, bs+t^*) - M(t^*)]}, \quad (y,s) \in [-1/a, 0] \times [-t^*/b, 0], \tag{2.2}$$

where $a = e^{-(q-m)M}$, $b = e^{-(2q-m-1)M}$. Clearly, $e^{-(m-1)M(t^*)} \leq \varphi_a \leq 1$, $\varphi_a(0,0) = 1$, $\frac{\partial \varphi_a}{\partial s} \geq 0$. By (1.1), it is easy to check with $k = \lambda e^{(p+m-2q)M}$ that

$$\begin{cases} (\varphi_a)_s = (m-1)\varphi_a(\varphi_a)_{yy} - k(m-1)(\varphi_a)^{\frac{p-1}{m-1}+1}, & (y,s) \in \left(-\dfrac{1}{a}, 0\right) \times \left(-\dfrac{t^*}{b}, 0\right), \\ (\varphi_a)_y(0,s) = (m-1)(\varphi_a)^{\frac{q-1}{m-1}}(0,s), \quad (\varphi_a)_y\left(-\dfrac{1}{a}, s\right) = 0, & s \in \left(-\dfrac{t^*}{b}, 0\right). \end{cases} \tag{2.3}$$

It is noted that positive functions a, b and k go to zero as $t^* \to T$ due to $q > m > 1$ and $p < 2q - m$. We claim that there exist positive constants C_1 and C_2 such that

$$C_1 \leq \frac{\partial \varphi_a}{\partial s}(0,0) \leq C_2 \tag{2.4}$$

holds for every a small.

The proof of (2.4) relies on the uniform boundedness of $\{\varphi_a\}$ and $\{(\varphi_a)_y\}$. Indeed, it is easy to see by (2.3) with $0 \leq \varphi_a \leq 1$ that $\{(\varphi_a)_y\}$ is also uniformly bounded. From the results for bounded solutions of porous medium type equations [13,14], $\{\varphi_a\}$ is equicontinuous on compact subsets of their common domain. Let $a_j = a(t_j^*)$ with $t_j^* \to T$, $j \to +\infty$. Passing to a subsequence if necessary, we have that $\varphi_{a_j} \to \varphi$ uniformly on compact subsets of $A = \{y \leq 0, s \leq 0\}$. The limit function φ is continuous with $\varphi(0,0) = 1$. Hence, for any $\varepsilon_0 \in (0,1)$, there exists a neighbourhood of (0,0), denoted by $U \subset A$, such that $\varphi > \varepsilon_0$ in U, and thus $\frac{\varepsilon_0}{2} \leq \varphi_{a_j} \leq 1$ on $\overline{U}$ for j large enough. By Schauder estimates [15],

$$\|\varphi_{a_j}\|_{C^{2+\alpha, 1+\alpha/2}(\overline{U})} \leq C. \tag{2.5}$$

The second inequality in (2.4) follows immediately.

If the first inequality in (2.4) fails, then there exists a sequence $a_j \to 0$ such that $\frac{\partial \varphi_{a_j}}{\partial s}(0,0) \to 0$. We proceed as before to obtain that $\varphi_{a_j} \to \varphi$, and the estimate (2.5) holds on compact subsets of $\{(y,s) : \varphi > 0\}$. Thus, we have $\varphi_{a_j} \to \varphi$ in $C^{2+\beta, 1+\beta/2}$ for some $\beta < \alpha$ satisfying $0 \leq \varphi \leq 1$, $\varphi(0,0) = 1$, $\frac{\partial \varphi}{\partial s} \geq 0$, and φ is a weak solution of

$$\begin{cases} \varphi_s = (m-1)\varphi\varphi_{yy}, \\ \varphi_y(0,s) = (m-1)\varphi^{\frac{q-1}{m-1}}(0,s) \end{cases} \tag{2.6}$$

in $\{y < 0\} \times (-\infty, 0]$. Define $z = \varphi_s$ to get

$$\begin{cases} z_s = (m-1)\varphi z_{yy} + (m-1)\varphi_{yy} z, \\ z_y(0,s) = (q-1)(\varphi^{\frac{q-m}{m-1}} z)(0,s) \geq 0, \\ z(0,0) = 0. \end{cases} \tag{2.7}$$

In the positivity set of φ, it follows by the Hopf lemma [16,17] that $z \equiv 0$, namely, φ is independent of s. Therefore, $\varphi = \varphi(y)$ satisfies

$$\begin{cases} 0 = \varphi_{yy}, & y < 0, \\ \varphi_y(0) = m - 1, & \varphi(0) = 1. \end{cases} \tag{2.8}$$

However, the problem (2.8) does not have bounded weak solutions.

In terms of w, it follows from $C_1 \le \frac{\partial \varphi_a}{\partial s}(0,0) \le C_2$ that

$$C_1 \le (m-1)\mathrm{e}^{-(2q-m-1)M} M_t(t^*) \le C_2.$$

Integrating the above inequality from t to T, we get

$$\log c(T-t)^{-\frac{1}{2q-m-1}} \le \max_{[0,1]} w(\cdot, t) = M(t) \le \log C(T-t)^{-\frac{1}{2q-m-1}}$$

with $c = [C_2(2q-m-1)/(m-1)]^{-\frac{1}{2q-m-1}}$ and $C = [C_1(2q-m-1)/(m-1)]^{-\frac{1}{2q-m-1}}$.

Next, treat the case $p = 2q - m$ in Theorems 1 and 2. For φ_a defined in (2.2), we know that φ_a satisfies (2.3) with $k = \lambda \mathrm{e}^{(p+m-2q)M} = \lambda$, and $\mathrm{e}^{-(m-1)M(t^*)} \le \varphi_a \le 1$, $\varphi_a(0,0) = 1$, $\frac{\partial \varphi_a}{\partial s} \ge 0$. We claim that (2.4) holds for this case also. Proceeding as before, we can obtain that the upper bound in (2.4) is true. If $\frac{\partial \varphi_a}{\partial s}(0,0) \ge C_1$ is false, then passing to a subsequence and using the Hopf lemma, we get a nontrivial solution of

$$\begin{cases} 0 = \varphi_{yy} - \lambda \varphi^{\frac{p-1}{m-1}}, & y < 0, \\ \varphi_y(0) = m - 1, & \varphi(0) = 1 \end{cases} \tag{2.9}$$

with $0 \le \varphi \le 1$. For $0 < \lambda < (m-1)(q-1)$ under $p = 2q - m$ required by Theorem 1, integrate $\varphi_y \varphi_{yy} - \lambda \varphi^{\frac{p-1}{m-1}} \varphi_y = 0$ on $(y, 0)$ to get an estimate for φ_y, and continue to reach a contradiction that the estimate $\varphi(y) \le 1 + \left(\frac{(m-1)[(m-1)(q-1)-\lambda]}{q-1}\right)^{\frac{1}{2}} y$ holds for $y \in (-\infty, 0]$. For $\lambda < 0$, we have $\varphi_{yy} \le 0$. However, bounded nontrivial concave functions do not exist in $(-\infty, 0]$. So, (2.4) is true in this case. Thereby we conclude

$$\log c(T-t)^{-\frac{1}{2q-m-1}} \le \max_{[0,1]} w(\cdot, t) \le \log C(T-t)^{-\frac{1}{2q-m-1}}.$$

Finally, consider the case $p > 2q - m$ with $\lambda < 0$ in Theorem 2. Define φ_a by (2.2) with $a = \mathrm{e}^{-\frac{p-m}{2}M}$, $b = \mathrm{e}^{-(p-1)M}$. Then $\mathrm{e}^{-(m-1)M(t^*)} \le \varphi_a \le 1$, $\varphi_a(0,0) = 1$, $\frac{\partial \varphi_a}{\partial s} \ge 0$, and

$$\begin{cases} (\varphi_a)_s = (m-1)\varphi_a(\varphi_a)_{yy} - \lambda(m-1)(\varphi_a)^{\frac{p-1}{m-1}+1}, & (y,s) \in \left(-\frac{1}{a}, 0\right) \times \left(-\frac{t^*}{b}, 0\right), \\ (\varphi_a)_y(0,s) = k(m-1)(\varphi_a)^{\frac{q-1}{m-1}}(0,s), \quad (\varphi_a)_y\left(-\frac{1}{a}, s\right) = 0, & s \in \left(-\frac{t^*}{b}, 0\right) \end{cases} \tag{2.10}$$

with $k = \mathrm{e}^{-(\frac{p+m}{2} - q)M}$. Clearly, a, b and k go to zero as $t^* \to T$ since $p > 2q - m$.

We can prove (2.4) for this case analogously. For instance, if $\frac{\partial \varphi_a}{\partial s}(0,0) \ge C_1$ is not true, passing to a subsequence and using the Hopf lemma, we obtain a nontrivial solution of

$$\begin{cases} 0 = \varphi_{yy} - \lambda \varphi^{\frac{p-1}{m-1}}, & y < 0, \\ \varphi_y(0) = 0, & \varphi(0) = 1 \end{cases} \tag{2.11}$$

with $0 \le \varphi \le 1$. This yields a bounded nontrivial concave function in $(-\infty, 0]$, a contradiction. From (2.4), we can establish the blow-up rate required in Theorem 2 for $p > 2q - m$. $\square$

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